Regularity of the Optimal Reward Operator

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Abstract

A player in a measurable gambling house Γ defined on a Polish state space X has available, for each $x \in X$, the collection $\Sigma(x)$ of possible distributions σ for the stochastic process $x_1, x_2, \ldots$ of future states. Suppose the object is to control the process so that it will lie in an analytic subset A of $H = X \times X \times \ldots$ and $M(A)(x)$, defined as $\sup\{\sigma(A) : \sigma \in \Sigma(x)\}$, corresponds to the player's optimal reward. If X is countable or if $\Gamma(x)$ is finite for each x, then $M(A)(x) = M^*(A)(x)$, where $M^*(E)(x)$ is defined for each subset E of H as the infimum of $M[\tau<\infty](x)$ taken over all Borel stopping times τ such that $E \subset [\tau < \infty]$. Under the same assumptions, $M^*(\cdot)(x)$ is a right-continuous capacity for each x.

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all measurable strategies σ at x. In the sequel, we shall frequently regard $\Sigma(x)$ as a set of probability measures on H, viz., the probability measures induced by measurable strategies at x.

The optimal reward operator M assigns to each universally measurable subset B of H the function M(B) defined on X by

$$M(B)(x) = \sup \{\sigma(B) : \sigma \in \Sigma(x)\}. \tag{1.1}$$

The question was raised in [10], and again in [7], as to whether the set function $M(\cdot)(x)$ has (outer) regularity properties like those of a measure. Partial answers to this question were obtained in [7]. In order to state the main result of [7], we must consider two topologies on H. The first of these, which we will denote by T_1, is the product topology on H when X is assigned the topology under which it is a Borel subset of a Polish space. The second, denoted by T_2, is the product topology on H when X is assigned the discrete topology. The words "Borel", "analytic", "coanalytic" when used to qualify subsets of H, will refer to the topology T_1, while the words "closed", "open", "clopen", "G_δ", "$G_{\delta\sigma}$" will be with respect to the topology T_2. In case of ambiguity, the relevant topology will be mentioned explicitly, thus we will write "T_1 - open", "T_2 - open" etc. (In a first reading, one might assume X is countable so that the two topologies are the same.)

The main result of [7] (Theorem 6.7), which will also be the point of departure for the present article, is as follows.

Theorem 1.1 If E is a countable intersection of Borel, open subsets of H, then

$$M(E)(x) = \inf \{M(O)(x) : E \subseteq O \text{ and } O \text{ is Borel, open}\} \tag{1.2}$$

for every $x \in X$.

1. Introduction and statement of results

Suppose X is a nonempty Borel subset of a Polish space and let $\mathcal{P}(X)$ be the collection of countably additive probability measures defined on the Borel subsets of X. Equip $\mathcal{P}(X)$ with its usual weak topology so that it too has the structure of a Borel subset of a Polish space (see, for example, chapter II of Parthasarathy [9] for information about the weak topology on $\mathcal{P}(X)$). An analytic gambling house is a mapping which assigns to each $x \in X$ a nonempty set $\Gamma(x) \subseteq \mathcal{P}(X)$ such that the set

$$\Gamma = \{(x,\gamma) \in X \times \mathcal{P}(X) : \gamma \in \Gamma(x)\} \tag{1.0}$$

is an analytic subset of $X \times \mathcal{P}(X)$. Starting at some initial state x, a player in the house Γ chooses a measurable strategy σ at x, which means a sequence $\sigma = (\sigma_0, \sigma_1, \ldots)$, where $\sigma_0 \in \Gamma(x)$ and, for $n = 1,2,\ldots$, σ_n is a universally measurable mapping from X^n to $\mathcal{P}(X)$ such that $\sigma_n(x_1,x_2,\ldots,x_n) \in \Gamma(x_n)$ for every $(x_1,x_2,\ldots,x_n) \in X^n$. In case each σ_n is analytically measurable from X^n to $\mathcal{P}(X)$ (i.e., σ_n is measurable when X^n is endowed with the σ-field generated by analytic subsets of X^n and $\mathcal{P}(X)$ is given its usual Borel σ-field) and $\sigma_n(x_1,x_2,\ldots,x_n) \in \Gamma(x_n)$ for every $(x_1,x_2,\ldots,x_n) \in X^n$, we say that σ is an analytically measurable strategy at x. Every measurable strategy σ determines a probability measure, also denoted by σ, on the Borel subsets of

$$H = X \times X \times \cdots .$$

This probability measure can be regarded as the distribution of the coordinate process $h_1,h_2,\ldots$, where h_1 has distribution σ_0 and h_{n+1} has conditional distribution $\sigma_n(x_1,x_2,\ldots,x_n)$ given $h_1 = x_1$, $h_2 = x_2,\ldots$, $h_n = x_n$. It is not hard to verify that if σ is a probability measure on the Borel subsets of H induced by a measurable strategy at x, then σ is already induced by an analytically measurable strategy at x. For $x \in X$, let $\Sigma(x)$ be the collection of

The main results of the present article can now be stated.

<u>Theorem 1.2</u> If E is a countable union of countable intersections of Borel, open subsets of H, then (1.2) holds for every $x \in X$.

Theorem 1.2 is the strongest result we are able to prove for a general, analytic gambling house. However, in some special cases, we can do much better.

<u>Theorem 1.3</u> If X is countable and E is an analytic subset of H, then (1.2) holds for every $x \in X$.

<u>Theorem 1.4</u> If $\Gamma(x)$ is finite for each $x \in X$ and E is an analytic subset of H, then (1.2) holds for each $x \in X$.

All of these theorems are concerned with the outer regularity of the set function $M(\cdot)(x)$. It should be mentioned that in contrast to the problem of outer regularity the problem of inner regularity over compact subsets in the topology T_1 is trivial. Indeed

$$M(B)(x) = \sup\{M(K)(x) : K \subseteq B \text{ and } K \text{ is } T_1\text{-compact}\} \tag{1.3}$$

holds for any universally measurable subset B of H and any $x \in X$. This is an immediate consequence of the inner regularity of a single probability measure ([8],p.61).

The theorems stated above can be regarded as giving conditions under which more general gambling problems can be approximated by the classical problems of

Dubins and Savage [4]. Suppose that, in the terminology of [4], the gambler's "utility function" is the indicator of a Borel subset A of X. Then the optimal return function U defined in [4] satisfies

$$U(x) = M([\tau_A < \infty])(x) \tag{1.4}$$

where τ_A is the hitting time of the set A. Now it was shown in [5] that a set $0 \subseteq H$ is Borel and open if and only if $0 = [\tau < \infty]$ for some Borel measurable stopping time τ defined on H. Furthermore, the problem of controlling the process $h_1, h_2, \ldots$ so that it will lie in $[\tau < \infty]$ is equivalent to controlling (h_1), (h_1,h_2), $(h_1,h_2,h_3), \ldots$ so as to reach the set of partial histories along which τ stops. Thus the problem of calculating M(0) is equivalent, after a change of coordinates, to that of finding a Dubins and Savage return function as in (1.4). Such functions have been well studied and can be calculated, at least in principle, by backward induction [4, Theorems 2.15.2 and 2.15.3].

The proof of Theorem 1.2 will be presented in section 2. Ideas from capacity theory will be used in section 3 to prove Theorems 1.3 and 1.4. The final section contains examples which help clarify certain aspects of the theory.

2. Squeezing $G_{\delta\sigma}$ sets

The main object of this section is to prove Theorem 1.2. Sets described in Theorem 1.2 will be called <u>special</u> $G_{\delta\sigma}$ sets, similarly sets appearing in Theorem 1.1 will be called <u>special</u> G_δ sets. Though it will not concern us here, let us mention in passing that it is an open question whether a Borel, G_δ subset of H is always a special G_δ set.

Our proofs of Theorems 1.3 and 1.4 will rely on an operator M^* which is defined, for every set $E \subseteq H$ and $x \in X$, by

$$(2.1) \qquad M^*(E)(x) = \inf \{M(O)(x) : E \subseteq O \text{ and } O \text{ is Borel, open}\}.$$

The next definition helps explain the title of this section. Let E be a universally measurable subset of H.

<u>Definition</u> (i) For $\epsilon > 0$ and $x \in X$, say that E is ϵ - <u>squeezable</u> at x if

$$M^*(E)(x) \le M(E)(x) + \epsilon.$$

(ii) Say that E is <u>squeezable</u> at x if $M^*(E)(x) = M(E)(x)$.

In view of this definition, Theorem 1.1 can be reformulated as stating that each special G_δ subset of H is squeezable at every x.

<u>Lemma</u> 2.1 For fixed x, the set function $M(\cdot)(x)$ and $M^*(\cdot)(x)$ are countably subadditive on the universally measurable subsets of H.

<u>Proof</u> : Straightforward using the countable subadditivity of each $\sigma \in \Sigma(x)$. □

All the sets occurring in the following lemmas are assumed to be at least universally measurable subsets of H unless there is a statement to the contrary.

Lemma 2.2 Suppose $E = \bigcup_{n=1}^{\infty} E_n$, and let $x \in X$. If $M(E)(x) = 0$ and each E_n is squeezable at x, then E is squeezable at x.

Proof: For each n, $M(E_n)(x) \le M(E)(x) = 0$. Since E_n is squeezable, $M^*(E_n)(x)=0$. Thus, by Lemma 2.1, $M^*(E)(x) = 0$. So E is squeezable at x. □

Lemma 2.3. If E is a special $G_{\delta\sigma}$ subset of H and $M(E)(x) = 0$, then E is squeezable at x.

Proof : Use Theorem 1.1 and Lemma 2.2. □

Lemma 2.4 If D is 2ϵ - squeezable at x and $M^*(N)(x) = 0$, then $D \cup N$ is 3ϵ - squeezable at x.

Proof : Let 0_1 and 0_2 be Borel, open sets satisfying:

$$M(0_1)(x) < M(D)(x) + 5\epsilon/2$$

$$M(0_2)(x) < \epsilon/2$$

$$D \subseteq 0_1,\ N \subseteq 0_2.$$

Then

$$\begin{aligned} M(0_1 \cup 0_2)(x) &\le M(0_1)(x) + M(0_2)(x) \\ &\le M(D)(x) + 3\epsilon \\ &\le M(D \cup N)(x) + 3\epsilon. \end{aligned}$$ □

Some additional notation is needed for the next few lemmas. Most of it is taken from Dubins and Savage [4]. Let X^* be the set of all finite sequences of elements of X. Regard X^* as the disjoint union of the sets X^n and give X^* the union topology when each X^n is endowed with the product topology obtained by giving X the topology under which it is a Borel subset of a Polish space. Then X^* will be a Borel subset of a Polish space. If E is a subset of H and $p \in X^*$, let

$$Ep = \{h \in H : ph \in E\}$$

where ph is the sequence in H consisting of the elements of p followed by those of h. For each $p = (x_1,x_2,\ldots,x_n) \in X^*$, let $l(p) = x_n$ be the last coordinate of p, and for each $h = (h_1,h_2,\ldots) \in H$ and $n = 1,2,\ldots$, let $p_n(h) = (h_1,h_2,\ldots,h_n)$.

<u>Lemma</u> 2.5 Let E be an analytic subset of H. Then the function $M(Ep)(l(p))$ is upper analytic on X^*, that is, for each real number a, the set

$$\{p \in X^* : M(Ep)(l(p)) > a\}$$

is analytic.

<u>Proof</u> : Indeed, the set above is the projection to the first coordinate of the analytic set

$$\{(p,\sigma) \in X^* \times \mathcal{P}(H) : \sigma \in \Sigma(l(p)) \text{ and } \sigma(Ep) > a\}$$

To see that the previous set is analytic, use the following facts:

(a) the map $p \to l(p)$ is Borel measurable,

(b) Σ, regarded as a subset of $X \times P(H)$, is analytic ([1] or [11]), and

(c) the map $\sigma \to \sigma(Ep)$ is upper analytic in σ and p ([3], p.119). □

Now let E be a Borel subset of H and $0 < \epsilon < 1$. Define $A_\epsilon \subseteq X^*$ by :

$$A_\epsilon = \{p \in X^* : M(Ep)(l(p)) > 1 - \epsilon\}, \tag{2.2}$$

and define a stopping time τ_ϵ on H by

$$(2.3)\qquad \tau_\epsilon(h) = \inf\{n : p_n(h) \in A_\epsilon\}$$
$$= \inf\{n : M(Ep_n(h))(h_n) > 1 - \epsilon\}$$

where we set $\inf(0) = +\infty$.

(For a gambler seeking to obtain an h in E, τ_ϵ is the first time the conditional chance can be made at least $1-\epsilon$.)

Lemma 2.6 (a) The set A_ϵ defined by formula (2.2) is analytic.
(b) The stopping time τ_ϵ defined by formula (2.3) is lower analytic, that is, for each positive integer n, the set $\{h: \tau(h) \le n\}$ is analytic.

Proof: (a) is an immediate consequence of Lemma 2.5 and (b) follows from (a). □

Lemma 2.7 $M(E \cap [\tau_\epsilon = \infty])(x) = 0$ for every x.

Proof: Suppose not. Then there is an x and $\sigma \in \Sigma(x)$ such that $E \cap [\tau_\epsilon = \infty]$ has positive measure under σ. By the Lévy zero-one law, there is an $h \in H$ and a positive integer n such that the conditional σ-probability of $E \cap [\tau_\epsilon = \infty]$ given $p_n(h)$ is greater than $1-\epsilon$. Let $\sigma[p_n(h)]$ denote the conditional strategy as in [4, p.11], so that this conditional probability statement can be written in the form

$$\sigma[p_n(h)]((E \cap [\tau_\epsilon = \infty])p_n(h)) > 1-\epsilon. \tag{2.4}$$

The set $[\tau_\epsilon = \infty]p_n(h)$ must be nonempty, because it is a superset of a set of positive measure. This implies that $p_i(h) \notin A_\epsilon$, $i=1,2,\ldots n$. For, if $(h_1,h_2,\ldots,h_i) \in A_\epsilon$ for some $i \leq n$, then $\tau_\epsilon(p_n(h)h')$ would be finite for all $h' \in H$ and, in consequence, $[\tau_\epsilon = \infty]p_n(h)$ would be empty.

On the other hand, by (2.4) and the fact that $\sigma[p_n(h)] \in \Sigma(h_n)$,

$$\begin{aligned} 1-\epsilon &< \sigma[p_n(h)]\ (Ep_n(h)) \\ &\leq M(Ep_n(h))(h_n), \end{aligned}$$

forcing $p_n(h) \in A_\epsilon$. This contradicts the observation of the previous paragraph.□

The next lemma is a version of the optimality equation. A more restricted version than the one appearing below was proved in [7] (Theorem 2.5). First we introduce some more notation. If τ is a stopping time and $\tau(h) < \infty$, we write h_τ for the $\tau(h)$-th coordinate of h and set $p_\tau = p_\tau(h) = (h_1,h_2,\ldots,h_\tau)$. For a set $B \subseteq H$, we denote by $Bp_\tau(h)$ the set

$$\{h' \in H : p_\tau(h)h' \in B\}.$$

When it is clear from the context what h is, we will suppress it and write Bp_τ instead of $Bp_\tau(h)$.

<u>Lemma</u> 2.8 Let B be an analytic subset of H and τ an analytically measurable stopping time such that $B \subseteq [\tau < \infty]$. Then the function $h \to M(Bp_\tau(h))(h_\tau)$ is analytically measurable and

$$M(B)(x) = \sup \{\textstyle\int_{[\tau<\infty]} M(Bp_\tau)(h_\tau)d\sigma : \sigma \in \Sigma(x)\}. \tag{2.5}$$

Proof: For the first assertion, observe that

$$M(Bp_\tau)(h_\tau) = M(Bp_n(h))(h_n) \text{ if } \tau(h) = n.$$

It is an easy consequence of Lemma 2.5 that the function $h \to M(Bp_n(h))(h_n)$ is upper analytic, from which it follows immediately that $h \to M(Bp_\tau)(h_\tau)$ is analytically measurable.

Let $\sigma \in \Sigma(x)$. Then

$$\begin{aligned}\sigma(B) &= \sigma(B \cap [\tau < \infty]) \\ &= \int_{[\tau < \infty]} \sigma[p_\tau](Bp_\tau) d\sigma \\ &\leq \int_{[\tau < \infty]} M(Bp_\tau)(h_\tau) d\sigma\end{aligned}$$

Take the sup over $\sigma \in \Sigma(x)$ to get

$$M(B)(x) \leq \sup_{\sigma\in\Sigma(x)} \int_{[\tau<\infty]} M(Bp_\tau)(h_\tau)\, d\sigma .$$

For the reverse inequality, let $\epsilon > 0$ and fix $x \in X$. It now suffices to find $\sigma^* \in \Sigma(x)$ such that

$$\sigma^*(B) \geq \sup_{\sigma\in\Sigma(x)} \int_{[\tau<\infty]} M(Bp_\tau)(h_\tau)\, d\sigma - \epsilon .$$

We now proceed to construct the strategy σ^*. First choose $\hat{\sigma} \in \Sigma(x)$ such that

$$\int_{[\tau<\infty]} M(Bp_\tau)(h_\tau)\, d\hat{\sigma} > \sup_{\sigma\in\Sigma(x)} \int_{[\tau<\infty]} M(Bp_\tau)(h_\tau)\, d\sigma - \frac{\epsilon}{2} .$$

Next, use the von Neumann selection theorem to find, for each n, an analytically measurable selector $\psi_n : H \to \mathcal{P}(H)$ for the set

$$\{(h,\sigma) \in H \times P(H) : \sigma \in \Sigma(h_n)\}$$

such that for every h

$$\psi_n(h)(Bp_n(h)) > M(Bp_n(h))(h_n) - \epsilon/2.$$

[The existence of such a ψ_n is shown in detail in Lemma 2.1 of [7].]

Let x^* be a fixed element of X and let $\psi : X \to \mathcal{P}(X)$ be an analytically measurable selector for the set Γ of (1.0). For each n, fix a Borel measurable mapping $\nu_n : \mathcal{P}(H) \times X^n \to \mathcal{P}(H)$ such that for each $\mu \in \mathcal{P}(H)$, $\nu_n(\mu, (x_1, x_2, \ldots x_n))$ is a regular conditional μ-distribution of $(h_{n+1}, h_{n+2}, \ldots)$ given $h_1 = x_1, h_2 = x_2, \ldots, h_n = x_n$. [The existence of ν_n is established in Lemma 2.2 of [6]]. Finally, for $\mu \in \mathcal{P}(H)$, let μ_0 denote the marginal distribution of the first coordinate h_1 under μ.

Here, at last, is the definition of σ^* :

$$\sigma_0^* = \hat{\sigma}_0$$

$$\begin{aligned}
\sigma_n^*(x_1, x_2, \ldots, x_n) &= \hat{\sigma}_n(x_1, x_2, \ldots x_n) \\
&\quad \text{if } 1 \le n < \tau(x_1, x_2, \ldots, x_n, x^*, x^*, \ldots), \\
&= (\nu_{n-i}(\psi_i(x_1, x_2, \ldots, x_i, x^*, x^*, \ldots), (x_{i+1}, x_{i+2}, \ldots, x_n)))_0 \\
&\quad \text{if } \tau(x_1, x_2, \ldots, x_n, x^*, x^*, \ldots) = i \le n \text{ and} \\
&\quad \nu_{n-i}(\psi_i(x_1, x_2, \ldots, x_i, x^*, x^*, \ldots), (x_{i+1}, x_{i+2}, \ldots x_n)))_0 \in \Gamma(x_n), \\
&= \psi(x_n) \\
&\quad \text{if } \tau(x_1, x_2, \ldots, x_n, x^*, x^*, \ldots) = i \le n \text{ and} \\
&\quad \nu_{n-i}(\psi_i(x_1, x_2, \ldots, x_i, x^*, x^*, \ldots), (x_{i+1}, x_{i+2}, \ldots, x_n)))_0 \notin \Gamma(x_n).
\end{aligned}$$

Then σ^* is measurable (indeed, σ^* is measurable with respect to the σ-field of C-sets, that is, the least family of sets containing the Borel sets and closed under the Souslin operation and complementation), so $\sigma^* \in \Sigma(x)$. Finally,

$$\begin{aligned}
\sigma^*(B) &= \int_{[\tau<\infty]} \sigma^*[p_\tau](Bp_\tau)\, d\sigma^* \\
&> \int_{[\tau<\infty]} M(Bp_\tau](h_\tau)\, d\sigma^* - \epsilon/2 \\
&= \int_{[\tau<\infty]} M(Bp_\tau)(h_\tau)\, d\hat{\sigma} - \epsilon/2
\end{aligned}$$

$$> \sup_{\sigma\epsilon\Sigma(x)} \int_{[\tau<\infty]} M(Bp_\tau)(h_\tau)\ d\sigma - \epsilon\ ,$$

as was to be proved. □

We now return to the Borel set $E \subseteq H$ of Lemma 2.6 and let A_ϵ, τ_ϵ be defined by formulas (2.2) and (2.3), respectively.

Lemma 2.9 For every x,

$$M([\tau_\epsilon < \infty])(x) \leq M(E \cap [\tau_\epsilon < \infty])(x) + \epsilon\ .$$

Proof : To simplify notation, set $\tau = \tau_\epsilon$. Notice that, on the set $[\tau < \infty]$, $[\tau < \infty]p_\tau = H$ and, by (2.3), $M(Ep_\tau)(h_\tau) > 1 - \epsilon$. Thus, for $\sigma \in \Sigma(x)$,

$$\begin{aligned}&\int_{[\tau<\infty]} M((E \cap [\tau<\infty])p_\tau(h_\tau)d\sigma \\ &= \int_{[\tau<\infty]} M(Ep_\tau)(h_\tau)d\sigma \\ &\geq (1 - \epsilon)\ \sigma([\tau < \infty])\ .\end{aligned}$$

Recall that, by Lemma 2.6(b), τ is lower analytic, hence $[\tau<\infty]$ is analytic, consequently, so is the set $E \cap [\tau<\infty]$. Now take the sup over $\sigma \in \Sigma(x)$ in the inequality above and apply Lemma 2.8. This yields

$$\begin{aligned}M(E \cap [\tau<\infty])(x) &\geq (1-\epsilon)\ M([\tau<\infty])(x) \\ &\geq M([\tau < \infty])(x) - \epsilon.\end{aligned}$$ □

Proof of Theorem 1.2: Let E be a special $G_{\delta\sigma}$ subset of H, and let $x_0 \in X$ be fixed. We must show that E is squeezable at x_0. So let $\epsilon > 0$, and suppose A_ϵ and τ_ϵ are defined by formulas (2.2) and (2.3), respectively. Now the set $[\tau_\epsilon < \infty]$ is analytic and open. By Corollary 4.4 in [7], there is a Borel set $B \supseteq [\tau_\epsilon < \infty]$ such that

$$M(B)(x_0) < M([\tau_\epsilon < \infty])(x_0) + \epsilon.$$

Now the analytic set $[\tau_\epsilon < \infty]$ can be separated from the analytic set B^c by an open set, viz., itself. So by Theorem 3.1 in [5], there is a Borel, open set 0 such that $[\tau_\epsilon < \infty] \subseteq 0 \subseteq B$. Consequently,

$$\begin{aligned} M(0)(x_o) &< M([\tau_\epsilon < \infty])(x_o) + \epsilon \\ &\leq M(E \cap [\tau_\epsilon < \infty])(x_o) + 2\epsilon \\ &\leq M(E \cap 0)(x_o) + 2\epsilon \ , \end{aligned}$$

where the second inequality is by virtue of Lemma 2.9. It follows that

$$M^*(E \cap 0)(x_o) \leq M(E \cap 0)(x_o) + 2\epsilon \ ,$$

so the set $E \cap 0$ is 2ϵ-squeezable at x_o.

On the other hand, 0^c is Borel, closed. Hence, by Corollary 3.2 of [5], 0^c is a special G_δ set, and, so $E \cap 0^c$ is a special $G_{\delta\sigma}$ set. Furthermore,

$$M(E \cap 0^c)(x_o) \leq M(E \cap [\tau_\epsilon = \infty])(x_o) = 0 \ ,$$

where we are using Lemma 2.7 to justify the equality above. So, by Lemma 2.3, $E \cap 0^c$ is squeezable at x_o. Consequently, $M^*(E \cap 0^c)(x_o) = 0$.

We can now apply Lemma 2.4 with $D = E \cap 0$ and $N = E \cap 0^c$, so that $E = D \cup N$ is 3ϵ-squeezable at x_0. Since ϵ is arbitrary, E is squeezable at x_o, completing the proof of Theorem 1.2. □

We shall now deduce certain properties of the operator M^* from Theorem 1.2.

Corollary 2.10. For every $x \in X$, the operator $M^*(\cdot)(x)$

(a) is monotone, that is,

(2.6) $$A \subseteq B \text{ implies } M^*(A)(x) \leq M^*(B)(x), \text{ and}$$

(b) has the "going up" property, that is,

(2.7) $$A_1 \subseteq A_2 \subseteq \ldots \text{ implies } M^*(\cup_n A_n)(x) = \lim_n M^*(A_n)(x) \ .$$

Proof : (a) is immediate from the definition of M^*. To prove (b), fix $x \in X$ and observe that $M(\cdot)(x)$ has the "going up" property along Borel sets, since each $\sigma \in \Sigma(x)$ has this property. Now let $A_1 \subseteq A_2 \subseteq \ldots$, and set $A = \cup_n A_n$ For each n, find a special G_δ set $G_n \supseteq A_n$ such that $M^*(A_n)(x) = M^*(G_n)(x)$. Define

$G_n' = \bigcap_{m \geq n} G_m$, so G_n' is again a special G_δ set, $G_n' \supseteq A_n$ and $M^*(A_n)(x) = M^*(G_n')(x)$. Moreover, $G_1' \subseteq G_2' \subseteq \ldots$ and $\cup_n G_n'$ is a special $G_{\delta\sigma}$ set. Finally,

$$\begin{aligned} M^*(A)(x) \leq M^*(\cup_n G_n')(x) = M(\cup_n G_n')(x) &= \lim_n M(G_n')(x) \\ &\leq \lim_n M^*(G_n')(x) = \lim_n M^*(A_n)(x), \end{aligned}$$

where the first equality is by virtue of Theorem 1.2 and the second because M has the "going up" property along Borel sets. Since $M^*(\cdot)(x)$ is monotone, $M^*(A)(x) \geq \lim_n M^*(A_n)(x)$ and so (2.7) is established. □

3. Capacitability

The properties of M^* listed in Corollary 2.10 are suggestive of those of a capacity. (A nice reference for capacity theory is Dellacherie [2].) We recall the definition of a capacity.

Let Y be a Hausdorff space. A *capacity* J on Y is a function from the power-set of Y into $[0,\infty]$ such that

(3.1) $\qquad A \subseteq B$ implies $J(A) \leq J(B)$,

(3.2) $\qquad A_1 \subseteq A_2 \subseteq \ldots$ implies $J(\cup_{n\geq 1} A_n) = \lim_n J(A_n)$,

(3.3) $\qquad K$ compact implies $J(K)<\infty$ and $J(K)=\inf\{J(0):K \subseteq 0 \text{ \& } 0 \text{ is open}\}$.

Property (3.3) is called the *right-continuity* property of J on compacts.

The next result is an immediate consequence of Corollary 2.10 and the definition of M^*.

Lemma 3.1. For every $x \in X$, $M^*(\circ)(x)$ is a capacity on H with respect to the topology T_2.

Lemma 3.1 is, however, not very useful in proving the squeezing property of T_1-analytic subsets of H, except when X is countable, in which case the two topologies coincide. It turns out that for the squeezing problem the capacity property of $M^*(\cdot)(x)$ with respect to the topology T_1 is more relevant. The next few lemmas will establish this in an important special case. An example will be presented in section 4 to show that, in general, $M^*(\cdot)(x)$ fails to be a capacity with respect to the topology T_1.

Let us call a gambling house Γ finite if $\Gamma(x)$ is finite for each $x \in X$. Note that the fortune space X is permitted to be infinite. For the next three lemmas, we will assume that X is a Borel subset of a Polish space and the gambling house Γ is finite and analytic.

First, there is some more notation to be explained. For a set $A \subseteq X$, we denote by A^∞ the product $A \times A \times \ldots$, and by A^m the product $A \times A \times \ldots \times A \times X \times X \times \ldots$ of m copies of A with copies of X, $m \geq 1$.

Lemma 3.2. Let $A_1 \supseteq A_2 \supseteq \ldots$ be Borel subsets of X, and let $A = \cap_{n\geq 1} A_n$. Then

$$M(A^\infty) = \lim_n M(A_n^\infty).$$

Proof: We first prove by induction on m that

(3.4) $$M(A^m) = \lim_n M(A_n^m) \ .$$

Fix $x_o \in X$. Choose a sequence $n_1 < n_2 < \ldots$ and $\gamma^* \in \Gamma(x_o)$ such that

$$M(A_{n_i}^1)(x_o) = \gamma^*(A_{n_i})$$

for all $i \geq 1$. Hence

$$\lim_n M(A_n^1)(x_o) = \lim_i M(A_{n_i}^1)(x_o) = \gamma^*(A) \leq M(A^1)(x_o) \ .$$

This establishes (3.4) for $m = 1$. Suppose (3.4) holds for $m = k$. Observe that for each $n \geq 1$

(3.5) $$M(A_n^{k+1})(x_o) = \sup_{\gamma \in \Gamma(x_o)} \int_{A_n} M(A_n^k)(x) \, d\gamma(x) \ .$$

The last formula can be obtained from Lemma 2.8 by setting $B = A_n^{k+1}$ and $\tau = 1$. So again there is a sequence $n_1 < n_2 < \ldots$ and $\gamma^* \in \Gamma(x_o)$ such that

$$M(A_{n_i}^{k+1})(x_o) = \int_{A_{n_i}} M(A_{n_i}^k)(x) \, d\gamma^*(x)$$

for all $i \geq 1$. So

$$\begin{aligned} \lim_n M(A_n^{k+1})(x_o) &= \lim_i M(A_{n_i}^{k+1})(x_o) \\ &= \lim_i \int_{A_{n_i}} M(A_{n_i}^k)(x) \, d\gamma^*(x) \\ &= \int_A M(A^k)(x) \, d\gamma^*(x) \\ &\leq \sup_{\gamma \in \Gamma(x_o)} \int_A M(A^k)(x) \, d\gamma(x) \\ &= M(A^{k+1})(x_o) \ , \end{aligned}$$

where the third equality uses the Dominated Convergence Theorem and the induction hypothesis and the inequality comes from formula (3.5) with A_n replaced by A. We have now established (3.4) for $m = k + 1$.

It now follows from (3.4) and Theorem 1.1 in [7] that

$$\begin{aligned}\lim_n M(A_n^\infty)(x_o) &= \lim_n \lim_m M(A_n^m)(x_o)\\ &= \lim_m \lim_n M(A_n^m)(x_o)\\ &= \lim_m M(A^m)(x_o)\\ &= M(A^\infty)(x_o).\end{aligned}$$ □

<u>Lemma</u> 3.3. Let $C_1 \supseteq C_2 \supseteq \ldots$ be Borel, closed subsets of H and let $C = \cap_{n\geq 1} C_n$. Then

$$M(C) = \lim_n M(C_n) \ .$$

<u>Sketch of proof</u>: The idea is quite simple, if a little cumbersome notationally. Given a Borel, closed subset D of H, the problem of maximizing the probability of getting into D is equivalent to maximizing the probability of staying forever in a Borel set of fortunes in a modified gambling problem, where the set of fortunes is essentially the set of partial histories in the original problem and the gambles in the new problem are the old gambles transported to the space of partial histories through an obvious map. Moreover, the new gambling problem is independent of the set D. The details of the construction can be found in section 6 of [7]. The proof is now completed by invoking Lemma 3.2. □

<u>Lemma</u> 3.4. For every $x \in X$, $M^*(\cdot)(x)$ has the "going down" property along T_1-closed sets, that is, if $C_1 \supseteq C_2 \supseteq \ldots$ are T_1-closed subsets of H, then

$$M^*(\cap_{n\geq 1} C_n)(x) = \lim_n M^*(C_n)(x) .$$

Proof : Since a T_1-closed subset of H is a special G_δ set, it follows from Theorem 1.2 (or 1.1) that $M^*(\cdot)(x)$ and $M(\cdot)(x)$ agree on T_1-closed subsets of H. The "going down" property of $M^*(\cdot)(x)$ now follows from Lemma 3.3. □

Theorem 3.5 Assume that X is countable or Γ is a finite gambling house. Then, for every $x \in X$, $M^*(\cdot)(x)$ is a capacity on H with respect to the topology T_1.

Proof: If X is countable, the assertion is contained in Lemma 3.1. Assume, then, that Γ is a finite gambling house. Corollary 2.10 ensures the monotone and "going up" properties of $M^*(\cdot)(x)$. We now verify the "right-continuity" property on compacts.

Let, then, K be a T_1-compact subset of H such that $M^*(K)(x) < a$. Choose a sequence of T_1-open sets $U_1 \supseteq U_2 \supseteq \ldots$ such that $\cap_{n\geq 1} U_n = K$. Using the regularity of the topology T_1 and the compactness of K, find, for each $n \geq 1$, a T_1-open set V_n such that $K \subseteq V_n \subseteq V_n^- \subseteq U_n$, where E^- will denote the T_1-closure of the set E. It now follows from Lemma 3.4 that

$$M^*(K)(x) = \lim_n M^*(V_1^- \cap V_2^- \cap \ldots \cap V_n^-)(x) .$$

So there is an m such that $M^*(U_1 \cap U_2 \cap \ldots \cap U_m)(x) < a$. Since $U_1 \cap U_2 \cap \ldots \cap U_m$ is a T_1 -open set containing K, this verifies the "right-continuity" property on compacts. □

The Capacitability Theorem connects the ideas developed in this section with the squeezing problem.

Theorem 3.6. Suppose, for every $x \in X$, $M^*(\cdot)(x)$ is a capacity on H with respect to the topology T_1. If E is a T_1-analytic subset of H, then $M^*(E)(x) = M(E)(x)$ for each $x \in X$.

Proof: Fix $x \in X$. Then, by formula (1.3),

$$M(E)(x) = \sup \{M(K)(x) : K \subseteq E \text{ and } K \text{ is } T_1\text{-compact}\}. \tag{3.6}$$

On the other hand, it follows from the Capacitability Theorem ([2], p.3) that

$$M^*(E)(x) = \sup \{M^*(K)(x) : K \subseteq E \text{ and } K \text{ is } T_1\text{-compact}\}. \tag{3.7}$$

But, by Theorem 1.1, $M(\cdot)(x)$ and $M^*(\cdot)(x)$ agree on T_1-closed subsets of H. So it follows from (3.6) and (3.7) that $M^*(E)(x) = M(E)(x)$. □

Theorems 1.3 and 1.4 now follow immediately from Theorem 3.5 and 3.6.

We conclude this section with the remark that the assertion in Theorem 3.5 pertaining to the case when X is countable and Theorem 1.3 remain valid even if we drop the condition that the gambling house Γ is analytic. This is seen by examining the proofs of the results that were used in proving the two theorems and observing that arguments establishing measurability of various objects are no longer needed since X is countable.

4. Examples

We present two examples in this section which will shed light on certain aspects of the theory developed in this article.

Example 1. Let $X = \{0,1\}$ and, as usual, let $H = X \times X \times \ldots$. Let S be a collection of probability measures defined on the Borel subsets of H. Define

$$\Lambda(B) = \sup \{\lambda(B) : \lambda \in S\}$$

for B a Borel subset of H, and set

$$\Lambda^*(E) = \inf \{\Lambda(0) : 0 \supseteq E \text{ and } 0 \text{ is open}\}$$

for arbitrary subsets E of H.

One could be misled by Theorems 3.5 and 1.3 into conjecturing that Λ^* is a capacity and that the regularity property corresponding to (1.2) holds. Here is an example from [10] which shows that both can fail when S is not of the form $\Sigma(x)$ for some gambling house Γ and $x \in X$.

For each $n \geq 1$, let $h^{(n)}$ be the element of H such that $h_i^{(n)} = 0$ for $i \leq n$ and $=1$ for $i > n$. Let

$$S = \{ \delta(h^{(n)}) : n = 1,2,\ldots \},$$

where $\delta(h)$ is point-mass at h. Take $E = A^{\infty}$, where $A = \{0\}$. Then, as is easy to verify, $\Lambda(E) = 0$ while $\Lambda^*(E) = 1$, so the regularity property corresponding to (1.2) fails. Furthermore, since for any open set $0 \supseteq E$, $\Lambda^*(0) = \Lambda(0) = 1$, the right-continuity property (3.3) fails to holds for Λ^*, hence Λ^* is not a capacity.

Example 2 This example shows that when the fortune space X is uncountable and the gambling house is not finite, the set function $M^*(\cdot)(x)$ need not be a capacity with respect to the topology T_1. In consequence, the methods of section 3, which enabled us to extend the squeezing property from special $G_{\delta\sigma}$ sets to all analytic sets, will not work for the general analytic gambling problem.

Let, for each $n \geq 1$, λ_n denote normalized Lebesgue measure on $[0, 1/n]$. Consider now the gambling problem where $X = [0,1]$, $\Gamma(0) = \{\lambda_n : n \geq 1\}$ and $\Gamma(x) = \{\delta(x)\}$ for $x \neq 0$. We claim that $M^*(\cdot)(0)$ does not possess the right-continuity property on compacts with respect to the topology T_1. To see this, take $K = \{0\} \times [0,1] \times [0,1] \times \ldots$. It is easy to see that $M^*(K)(0) = 0$, but for any T_1 - open set $U \supseteq K$, $M^*(U)(0) = 1$.

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